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Optimal Vibration Quenching for an Euler–Bernoulli Beam

J. M. Sloss

Department of Mathematics, University of California, Santa Barbara, California 93106

Department of Mechanical and Environmental Engineering and Department of Mathematics, University of California, Santa Barbara, California 93106

and

C. C. Kao

Department of Mathematics, Santa Barbara City College, Santa Barbara, California 93109

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The quenching of the vibration of an Euler–Bernoulli beam under tension with general linear homogeneous boundary conditions is studied using a distributed control. A method for determining the control that quenches a finite number of modes is given and it is shown that the method can be extended theoretically to determine a control to quench all modes of the vibration. In general there is more than one control that can be used to quench the same modes. It is shown that of all controls that quench specified modes of vibration at a given time and are square integrable the method described yields the unique control whose mean square is minimum. A method is given for determining how many modes are sufficient to be quenched if the residual position and velocity of the beam are both to remain within a restricted band after the control is removed. Numerical results are given in graphical form. © 1999 Academic Press

1. INTRODUCTION

Beams are commonly used as elements in the construction of a large number of structures such as spacecraft, large space structures, and robots. All these structures have flexible extensions and/or platforms which are

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made as light and slender as possible due to heavy penalties attached to excessive weight. Such slender elements lack the necessary damping properties to be able to function effectively under dynamic loads and may require a control mechanism to damp out excessive vibrations and to improve their performance characteristics. Moreover, the destabilizing effects of dynamic loads need to be counteracted to avoid failure under operational conditions.

When considering undesirable structural vibratory motion initiated by a disturbance, it is natural to introduce a control mechanism. An effective form of vibration suppression is to eliminate, when possible, the vibration of specific modes at a preassigned time and then turn the control off.

Jayasuriya and Choura [1] considered active control of the vibratory motion of a beam using a single fixed point actuator. The control is applied for the time t, $0 \leq t \leq T$, T some preassigned final time after which the control is switched off. For the case when the beam is simply supported and initial conditions are expressed as a linear combination of a finite number of modes, conditions are given that will ensure the control will quench all modes.

Trautt [2] derived an inverse dynamics numerical algorithm for active vibration quenching of structures. His algorithm uses frequency domain techniques to compute an input function needed to produce a desired response at a particular degree of freedom.

A method is given herein for determining a distributed control that will quench a finite number of special modes at time T of an Euler−Bernoulli beam which satisfies homogeneous boundary conditions and general initial conditions. An example shows there are in general infinitely many controls that will perform the quenching of the same modes. A particular procedure is presented that leads to a quenching control of individual modes. This is shown to be extendable to determining a control that will completely quench the vibration. Moreover, it is shown that this extended control is the unique control that will use the least expenditure of the control force.

In practice it is not possible to quench all modes. An example is given to show that if N is any given positive integer and the first N modes are quenched at time T with the control being turned off at time T, then the residual modal deflection can be very large for time greater than T. It is shown that if a mild restriction on the initial conditions is made, then it is possible to choose enough quenched terms to insure that for all time the deflection, as well as the velocity, can be made as small as any preassigned value. This observation is applied to a beam with various boundary conditions, and numerical results are given for the quenching of the first few modes of a cantilevered beam.

2. PROBLEM FORMULATION

Consider the flexural vibration of a controlled beam whose transverse displacement $w(x, t)$ satisfies

$$m(x)w_{tt} + L[w] = u(x, t) \qquad 0 < x < l, 0 < t < T, \tag{1}$$

in which

$$L[w] = (EI(x)w_{xx})_{xx} - (P(x)w_x)_x,$$

where m is the mass per unit length of the beam, P is the axial tensile force, E is Young's modulus of the material, I is the moment of inertia, $u(x, t)$ is the applied control force per unit length, l is the length of the beam, and T is the terminal time. In the considerations herein it will be assumed that $u(x, t) \equiv 0$ when $t \geq T$.

In addition w is assumed to satisfy the initial conditions

$$w(x, 0) = f(x), \qquad w_t(x, 0) = g(x), \tag{2}$$

where $f \in H^1(0, l)$, $g \in L^2(0, l)$, and homogeneous boundary conditions

$$\begin{aligned} B_j[w(x, t; u)]_{x=0} &= 0 \qquad j = 1, 2 \\ B_j[w(x, t; u)]_{x=l} &= 0 \qquad j = 3, 4. \end{aligned} \tag{3}$$

The state variable $w(x, t)$ is said to be "quenched at $t = T$" provided $w(x, T) = 0$ and $w_t(x, T) = 0$. Let

$$U_{\text{ad}}^0 = \{u \mid u(x, t) \in L^2[0, T; L^2(0, l)]\}. \tag{4}$$

If for $u \in U_{\text{ad}}^0$ the corresponding state variable $w(x, t; u)$ is quenched at $t = T$, then u will be called a *quencher for $w(x, t; u)$ at $t = T$*.

Quenchers for w at $t = T$ are not in general unique.

EXAMPLE. Consider

$$w_{tt} + w_{xxxx} = u \qquad 0 < x < l, 0 < t < T$$

with boundary conditions

$$\begin{aligned} w(0, t) &= 0, \qquad w(l, t) = 0 \\ w_x(0, t) &= 0, \qquad w_x(l, t) = 0 \end{aligned}$$

and initial conditions

$$w(x, 0) = -x^2(x - l)^2, \qquad w_t(x, 0) = 0$$

with the control taken to be of the form

$$u_r(x,t) = q_{r,tt} + q_{r,xxxx}$$

where

$$q_r(x,t) = \left[3\left(1 - \frac{t}{T}\right)^2 - 2\left(1 - \frac{t}{T}\right)^3\right](t^r - 1)x^2(x - l)^2, \qquad r \geq 2.$$

The corresponding state variable is given by

$$w(x,t;u_r) = q_r(x,t), \qquad r \geq 2$$

which is quenched at $t = T$. Thus $u_r(x,t)$ is a quencher for w at $t = T$. This holds for $r \geq 2$ and hence quenchers for w at $t = T$ are not unique.

Let

$$U_{\text{ad}} = \left\{u \middle| u \in U_{\text{ad}}^0 \text{ and } u \text{ is a quencher for } w(x,t;u) \text{ at } t = T\right\}. \tag{5}$$

It is therefore reasonable not only to require that u be a quencher for w at $t = T$ but to require some other quality of u be satisfied. To this end introduce the measure of applied force

$$E(u) = \int_0^T \int_0^l m(x)u^2(x,t)\,dx\,dt \qquad u \in U_{\text{ad}}. \tag{6}$$

The objective is then to find $u^* \in U_{\text{ad}}$ such that

$$E(u^*) = \min_{u \in U_{\text{ad}}} E(u). \tag{7}$$

From convexity it follows that there is at most one $u^* \in U_{\text{ad}}$ for which the minimum of $E(u)$ is obtained. This u^* will be called the *optimal quencher*.

3. CONSTRUCTION OF THE OPTIMAL QUENCHER

We shall assume that L has a complete set of orthonormal eigenfunctions $\{\varphi_n\}$, $\varphi_n \in L^2[0,l]$, with

$$L[\varphi_n] = \lambda_n^2 m(x)\varphi_n(x) \qquad 0 < \lambda_1 < \lambda_2 < \cdots \tag{8}$$

in which the inner product is denoted by

$$\langle \alpha, \beta \rangle = \int_0^l m(x)\alpha(x)\beta(x)\,dx. \tag{9}$$

LEMMA 1. *If $u(t)$ is the control*

$$u(t) = \sigma \cos \lambda t + \mu \sin \lambda t \qquad \lambda > 0 \tag{10}$$

in which

$$\sigma = a\left[(2 \sin^2 \lambda T) A + (2\lambda T - \sin 2\lambda T) B\right] \tag{11}$$

and

$$\mu = -a\left[(2\lambda T + \sin 2\lambda T) A + (2 \sin^2 \lambda T) B\right] \tag{12}$$

with

$$a = \frac{\lambda^2}{\sin^2 \lambda T - \lambda^2 T^2}, \tag{13}$$

then the solution $z(t)$ of

$$\dot{z} + \lambda^2 z = u(t), \qquad z(0) = A, \qquad \dot{z}(0) = \lambda B \tag{14}$$

satisfies

$$z(T) = 0, \qquad \dot{z}(T) = 0. \tag{15}$$

Proof. The general solution of (14) is given by

$$z(t) = A \cos \lambda t + B \sin \lambda t + \frac{1}{\lambda} \int_0^t \sin \lambda(t - s) u(s)\, ds. \tag{16}$$

By a direct and lengthy calculation (see Kao [3]) it is shown that

$$z(T) = 0 \quad \text{and} \quad \dot{z}(T) = 0$$

which is the derived result.

LEMMA 2. *If $\hat{u}(t) \neq u(t)$ on a set of positive measure is any other control of* (14) *such that*

$$\hat{z}(T) = 0, \qquad \dot{\hat{z}}(T) = 0,$$

then

$$\int_0^T \hat{u}^2(t)\, dt > \int_0^T u^2(t)\, dt.$$

For the proof, see, for example, Barnett and Cameron [4].

LEMMA 3. *Let $F(x) \in H_0^4(0, l)$, $EI(x) \in C^2[0, l]$, $P(x) \in C^1[0, l]$, $m(x) \in C^4[0, l]$, $m(x) \geq m_0 > 0$,*

$$L[\varphi_n] = \lambda_n^2 \varphi_n, \qquad \langle \varphi_n, \varphi_n \rangle = 1.$$

Then

$$\langle F, \varphi_n \rangle \leq \lambda_n^{-2} M, \qquad M \textit{ independent of } n. \tag{17}$$

Proof. Note that

$$\begin{aligned}
\lambda_n^2 \langle F, \varphi_n \rangle &= \lambda_n^2 \int_0^l m(x) F(x) \varphi_n(x)\, dx \\
&= \int_0^l F(x) L[\varphi_n]\, dx \\
&= \int_0^l m^{-1/2} \left[(EI(F)_{xx})_{xx} - (P(F)_x)_x \right] m^{1/2} \varphi_n\, dx \\
&\leq M
\end{aligned}$$

where

$$M = \left\| m^{-1/2} \left[(EI(F)_{xx})_{xx} - (P(F_x))_x \right] \right\|_{L^2} \tag{18}$$

and the Cauchy–Schwarz inequality has been used together with the fact that

$$\langle \varphi_n, \varphi_n \rangle = \int_0^l m \varphi_n^2\, dx = 1.$$

The result follows.

If $w(x, t)$ is the solution of (1) and

$$w(x, t) = \sum_{n=1}^{\infty} z_n(t) \varphi_n(x) \tag{19}$$

is the corresponding eigenfunction expansion, then from the differential equation we obtain

$$\ddot{z}_n + \lambda_n^2 z_n = u_n(t) \tag{20}$$

in which

$$u_n(t) = \int_0^l [u(x, t)/m(x)] m(x) \varphi_n(x)\, dx = \langle u/m, \varphi_n \rangle \tag{21}$$

and if the initial conditions

$$f(x) = \sum_{n=1}^{\infty} f_n \varphi_n(x), \qquad g(x) = \sum_{n=1}^{\infty} g_n \varphi_n(x) \tag{22}$$

with

$$f_n = \langle f, \varphi_n \rangle, \qquad g_n = \langle g, \varphi_n \rangle, \tag{23}$$

the modal problem becomes one of solving (20) with

$$z_n(0) = f_n, \qquad \dot{z}_n(0) = g_n \tag{24}$$

and

$$z_n(T) = 0, \qquad \dot{z}_n(T) = 0. \tag{25}$$

The solution of this is given by Lemma 1 with $u \to u_n$, $\sigma \to \sigma_n$, $\mu \to \mu_n$, $\lambda \to \lambda_n$, $a \to a_n$.

4. QUENCHING ALL MODES

The question of interest then becomes, given quenching at the modal level, can we be assured of having quenching in the general setting. Stated more precisely, can we show that

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t)\, \varphi_n(x) \tag{26}$$

does indeed define a function in $L^2[0,T; L^2(0,l)]$ when $u_n(t)$ is determined from

$$u_n(t) = \sigma_n \cos \lambda_n + \mu_n \sin \lambda_n t \tag{27}$$

of Lemma 1? The answer is yes provided the conditions of Theorem 1 are satisfied.

LEMMA 4. *Assume the hypothesis of Lemma 3 for $F = f$ and $F = g$ are satisfied, $T \geq T_0 \geq \lambda_1^{-1} > 0$. Then*

$$u_n^2(t) \leq \frac{M_1}{\lambda_n^2}, \qquad M_1 \textit{ a constant independent of } n. \tag{28}$$

Proof. By the Cauchy–Schwarz inequality

$$u_n^2 \leq \left(\sigma_n^2 + \mu_n^2 \right)$$

where

$$\sigma_n^2 \leq a_n^2 \left[\left(2 \sin^2 \lambda_n T\right)^2 + (2\lambda T - \sin 2\lambda T)^2 \right] \left(A_n^2 + B_n^2 \right),$$

$$\mu_n^2 \leq a_n^2 \left[(2\lambda_n T + \sin 2\lambda_n T)^2 + \left(2 \sin^2 \lambda_n T\right)^2 \right] \left(A_n^2 + B_n^2 \right).$$

Moreover, since $0 < \lambda_1 < \lambda_2 < \lambda_3 < \cdots$

$$T^2 - \lambda_n^{-2} \sin^2 \lambda_n T \geq T_0^2 - \lambda_n^{-2} \sin^2 \lambda_n T_0 \geq T_0^2 - \lambda_1^{-2} > 0$$

it follows that

$$|a_n| = \left| \frac{\lambda_n^2}{\lambda_n^2 T^2 - \sin^2 \lambda_n T} \right| \leq \frac{1}{T_0^2 - \lambda_1^{-2}}$$

with $a_n \to T^{-2}$ when $n \to \infty$. Hence

$$\begin{aligned} u_n^2 &\leq \frac{2}{T_0^2 - \lambda_1^{-2}} \left[\sin^4 \lambda_n T + 4\lambda_n^2 T^2 + \sin^2 2\lambda_n T \right] \left[A_n^2 + B_n^2 \right] \\ &\leq \frac{2M^2}{T_0^2 - \lambda_1^{-2}} \left[\frac{\sin^4 \lambda_n T}{\lambda_n^4} + \frac{4T^2}{\lambda_n^2} + \frac{\sin^2 2\lambda_n T}{\lambda_n^4} \right] \end{aligned}$$

by Lemma 3. It follows that

$$u_n^2 \leq \frac{4M^2}{T_0^2 - \lambda_1^{-2}} \left[\frac{1}{\lambda_n^4} + \frac{2T^2}{\lambda_n^2} \right] \leq \frac{M_1}{\lambda_n^2}$$

since $T_0 > \lambda_1^{-1}$ and

$$M_1 = \frac{4M^2}{T_0^2 - \lambda_1^{-2}} \left[\frac{1}{\lambda_1^2} + 2T^2 \right]. \tag{29}$$

Remark. With the aid of (28) and the assumption $\lambda_k > k^{\beta}$, $\beta > \frac{1}{2}$, it follows that (26) does indeed define a function in $L^2[0, T; L^2(0, l)]$ and hence the answer to the question formulated before Lemma 4 is confirmed.

THEOREM 1. *If the hypotheses of Lemmas 3 and 4 hold as well as*

$$\sum_{n=1}^{\infty} \lambda_n^{-2} < M_2 < \infty,$$

for some constant M_2, then

$$\sum_{n=1}^{\infty} u_n(t) \varphi_n(x)$$

does indeed define a function $u(x, t) \in L^2[0, T; L^2(0, l)]$ that quenches the solution $w(x, t)$ for all modes.

Proof. By Lemma 4

$$u_n^2 \leq \frac{1}{\lambda_n^2}$$

and by the hypothesis

$$\sum_{n=1}^{\infty} u_n^2(t) \leq M_2.$$

It follows from the Riesz–Fisher theorem that

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t)\,\varphi_n(x)$$

defines a function in $u(x,t) \in L^2[0,T; L^2(0,l)]$ which was constructed to quench each mode of the solution and hence $u \in U_{\text{ad}}^0$.

As shown earlier this function is not in general unique. However, the following holds:

THEOREM 2. *The u constructed above is the unique element of* U_{ad}^0 *such that*

$$E(u) = \min_{\hat{u} \in U_{\text{ad}}} E(\hat{u}). \tag{30}$$

Proof. From the definition of $E(\hat{u})$ given in (6) we have for $\hat{u}(t) \neq u(t)$ on a set of positive measure

$$\begin{aligned}
E(\hat{u}) &= \int_0^T \int_0^l m(x)\hat{u}^2(x,t)\,dx\,dt \\
&= \int_0^T \sum_{n=1}^{\infty} \hat{u}_n^2(t)\,dt \\
&> \int_0^T \sum_{n=1}^{\infty} u_n^2(t)\,dt \qquad \text{(from Lemma 2)} \\
&= \int_0^T \int_0^l m(x)u^2(x,t)\,dx\,dt \\
&= E(u)
\end{aligned}$$

which is the result.

In practice only a finite number of modes can be quenched. If the first N modes of the solution $w(x,t)$ are quenched at terminal time T, the question arises: is it reasonable to assume that the N mode quenched solution will not deviate from the totally quenched solution by a significant amount? The following example shows that without further restrictions, a meaningful answer to this question cannot be given.

Note that if the control $u(x,t)$ quenches all modes at $t = T$ and if the control is set equal to zero for $t \geq T$, then the solution will remain zero for $t \geq T$.

In the following example it will be shown that even if the first N modes (no matter how large N) are quenched then the resulting solution can deviate by a large amount from the fully quenched solution.

EXAMPLE. Consider

$$w_{tt} + w_{xxxx} = u(x,t), \qquad 0 < x < \pi, 0 < t < T$$

with simply supported boundary conditions

$$w(0,t) = w_{xx}(0,t) = 0,$$
$$w(\pi,t) = w_{xx}(\pi,t) = 0$$

and initial conditions

$$w(x,0) = f(x) = Q\sin(N+1)x,$$
$$w_t(x,0) = g(x) = 0.$$

Let the control $u(x,t)$ be chosen so that the first N modes of $w(x,t;u)$ are quenched at $t = T$ and $u(x,t)$ is optimal. Note that

$$\lambda_n = n^2, \qquad \varphi_n(x) = \sin nx$$

and

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t)\varphi_n(x)$$

with u_n given in Lemma 1 by

$$u_n(t) = \sigma_n \cos \lambda_n t + \mu_n \sin \lambda_n t$$
$$\sigma_n = 2a_n f_n \sin \lambda_n^2 T, \qquad \mu_n = -a_n f_n(2\lambda_n T + \sin \lambda_n T),$$
$$a_n = \frac{\lambda_n^2}{\sin^2 \lambda_n T - \lambda_n^2 T}$$

since $g_n = 0$. Note also that $f_n = 0$ if $n \neq N+1$ and that $f_{N+1} = 2Q$, from which it is seen that

$$u_n(t) = 0 \qquad \text{if } n \neq N+1.$$

It follows that the optimal quencher that quenches the first N modes is given by

$$u(x,t) = u_{N+1}(t)\,\varphi_{N+1}(x)$$

in which

$$u_{N+1}(t) = f_{N+1}F(t),$$

where

$$F(t) = a_{N+1}\big[2\sin^2 \lambda_{N+1}T\cos\lambda_{N+1}t - (2\lambda_{N+1}T + \sin\lambda_{N+1}T)\sin\lambda_{N+1}t\big].$$

The solution is thus

$$w(x,t) = u^{(1)}_{N+1}(t)\,\varphi_{N+1}(x) \qquad \text{for } 0 \le t \le T,$$

$$w(x,t) = u^{(2)}_{N+1}(t)\,\varphi_{N+1}(x) \qquad \text{for } t \ge T,$$

where

$$u^{(1)}_{N+1}(t) = f_{N+1}G(t),$$

in which

$$G(t) = \cos\lambda_{N+1}t + \frac{1}{\lambda_{N+1}}\int_0^t \sin\lambda_{N+1}(t-s)F(s)\,ds$$

and

$$u^{(2)}_{N+1}(t) = f_{N+1}H(t),$$

in which

$$H(t) = G(t)\cos\lambda_{N+1}t + \lambda^{-1}_{N+1}\dot{G}(t)\sin\lambda_{N+1}t.$$

However, note that if $w_0(x,t)$ is the totally quenched solution at $t = T$, then

$$\begin{aligned}\|w(\cdot,t) - w_0(\cdot,t)\|^2_{L^2} &= \int_0^\pi \big[w(x,t) - w_0(x,t)\big]^2\,dx\\ &= \int_0^\pi w^2(x,t)\,dx = 2\pi Q^2\tilde{P}^2(t) \qquad \text{for } t \ge T,\end{aligned}$$

where $\tilde{P}^2(t) > 0$ and does not depend on n. Hence given any N it is possible to quench the first N modes (even optimally) and yet have the resulting solution deviate from the fully quenched solution by a very large amount (choose Q large).

In the following it will be shown that if in addition to a control quenching the first N modes, it is also assumed that the part of the initial conditions corresponding to the unquenched modes is small, then the solution remains small. Since quenching can be done practically for at most a finite number of modes this becomes of practical importance.

DEFINITION. If for time $t \geq T$ the control u is set equal to zero, then

$$\int_0^l m(x)\left[w^2(x,t) + w_t^2(x,t)\right] dx$$

will be called the residual.

THEOREM 3. *Given* $\varepsilon > 0$ *and*

$$\sum_{n=1}^{\infty} \left[\left(1 + \lambda_n^2\right)\left(f_n^2 + \lambda_n^{-2} g_n^2\right)\right] < \infty, \tag{31}$$

then there is an N *and a control* $u_N(x,t)$ *that will quench the first* $N - 1$ *modes of* w *at* $t = T$ *and if the control is set equal to zero, then the residual will be less than* ε *for* $t \geq T$,

$$\int_0^l m(x)\left[w^2(x,t;u_N) + w_t^2(x,t;u_N)\right] dx < \varepsilon \qquad \text{for } t \geq T. \tag{32}$$

Proof. From assumption (31) it is possible to choose N so large that

$$\sum_{n=N+1}^{\infty} \left[\left(1 + \lambda_n^2\right)\left(f_n^2 + \lambda_n^{-2} g_n^2\right)\right] < \frac{\varepsilon}{2}.$$

Let

$$\bar{u}_N(x,t) = \sum_{n=1}^{N} u_n(t)\,\varphi_n(x)$$

be a control that quenches the first $N - 1$ modes and let

$$u_N(x,t) = \bar{u}_N(x,t) \qquad 0 \leq t \leq T,$$
$$u_N(x,t) = 0 \qquad T \leq t$$

which also quenches the first N modes of w. Let the solution be given by

$$w(x,t;u_N) = \sum_{n=1}^{\infty} z_n(t)\,\varphi_n(x).$$

Then

$$z_n(T) = \dot{z}_n(T) = 0$$

with

$$z_n(t) = z_n^h(t) + z_n^p(t)$$

in which

$$z_n^h(t) = f_n \cos \lambda_n t + \left(\frac{g_n}{\lambda_n}\right) \sin \lambda_n t$$

$$z_n^p(t) = \frac{1}{\lambda_n} \int_0^t \sin \lambda_n (t-s) \langle u_N(\cdot, t), \varphi_n \rangle (s)\, ds \qquad (n = 1, 2, \dots)$$

$$= \frac{1}{\lambda_n} \int_0^T \sin \lambda_n (t-s) \langle u_N(\cdot, t), \varphi_n \rangle (s)\, ds, \qquad (n = 1, 2, \dots)$$

since $u_N(x, t) \equiv 0$ for $t \geq T$. It follows that

$$z_n^p(t) = \begin{cases} \dfrac{1}{\lambda_n} \displaystyle\int_0^T \sin \lambda_n (t-s) u_n(s)\, ds, & n = 1, 2, \dots, N, \\ 0 & n \geq N+1. \end{cases}$$

Moreover, for $t \geq T$

$$m(x) w_{tt} + L[w] = 0 \qquad t > T, 0 < x < l$$

$$w(x, T) = \sum_{n=N+1}^{\infty} z_n^h(T) \varphi_n(x),$$

$$w_t(x, T) = \sum_{n=N+1}^{\infty} \dot{z}_n^h(T) \varphi_n(x)$$

$$B_j[w]_{x=0} = 0, \qquad j = 1, 2; \qquad B_j[w]_{x=l} = 0 \qquad j = 3, 4;$$

hence

$$\int_0^l m(x) \left[w^2(x, t; u_N) + w_t^2(x, t; u_N) \right] dx$$

$$= \sum_{n=N+1}^{\infty} \left[\left(z_n^h\right)^2(t) + \left(\dot{z}_n^h\right)^2(t) \right]$$

$$\leq 2 \sum_{n=N+1}^{\infty} \left[f_n^2 + \lambda_n^{-2} g_n^2 + \lambda_n^2 f_n^2 + g_n^2 \right] < \varepsilon$$

and the proof is complete.

THEOREM 4. *If in addition to the assumptions of Theorem* 3,

(1) $L[f] \in L^2(0,l)$, $L[g] \in L^2(0,l)$,

(2) *f and g satisfy homogeneous adjoint boundary conditions,*

then

$$\sum_{n=1}^{\infty} \left(1+\lambda_n^2\right)\left(f_n^2+\lambda_n^{-}g_n^2\right) \leq \| L[f] \|_{L^2}^2 A(1) + \| L[g] \|_{L^2}^2 B(1), \quad (33)$$

where

$$A(N) = C \sum_{n=N}^{\infty} \left(1+\lambda_n^2\right)\lambda_n^{-4},$$

$$B(N) = C \sum_{n=N}^{\infty} \left(1+\lambda_n^2\right)\lambda_n^{-6},$$

in which

$$C = \left[\max_{0 \leq x \leq l} m^{-1}(x)\right]^{1/2}.$$

Proof. Since f and g satisfy homogeneous adjoint boundary conditions and φ_n satisfies homogeneous boundary conditions, it follows that the bilinear forms

$$\beta[f,\varphi_n] = f[EI\varphi_{n,xx}]_x - f_x[EI\varphi_{n,xx}] + [f_{xx}EI]\varphi_{n,x}$$
$$- [f_{xx}EI]_x\varphi_n - fP\varphi_{n,x} + f_xP\varphi_n$$

are equal to zero at $x = 0$ and at $x = l$. A similar result holds when f is replaced by g.

Since

$$L[\varphi_n] = \Omega_n^4 m(x)\varphi_n(x) \qquad \Omega_n > 0,$$

then

$$f_n = \int_0^l m(x)f(x)\varphi_n(x)\,dx$$

$$= \lambda_n^{-2}\int_0^l fL[\varphi_n]\,dx \qquad \left(\lambda_n = \Omega_n^2\right)$$

$$= \lambda_n^{-2}\left\{\beta[f,\varphi_n]_{x=0}^{x=l} + \int_0^l L[f]\varphi_n\,dx\right\}$$

$$= \lambda_n^{-2}\int_0^l \left(m^{-1/2}L[f]\right)\left(m^{1/2}\varphi_n\right)dx. \quad (34)$$

Making use of the Cauchy–Schwarz inequality and the fact that the φ_n are orthonormal it follows that

$$|f_n| \leq \lambda_n^{-2} \| m^{-1/2} L[f] \|_{L^2} \langle \varphi_n, \varphi_n \rangle = \lambda_n^{-2} \| m^{-1/2} L[f] \|_{L^2}.$$

In a similar way

$$|g_n| \leq \lambda_n^{-2} \| m^{-1/2} L[g] \|_{L^2}$$

and the result follows.

COROLLARY. *If in addition to the assumptions of Theorem* 3,

(1) $L[f] \in L^2(0,l)$, $L[g] \in L^2(0,l)$,

(2) $|\beta[f, \varphi_n]_{x=0}^{l}| \leq \lambda_n^{1/2} C_f$ *for* $n \geq N_0$,

(3) $|\beta[g, \varphi_n]_{x=0}^{l}| \leq \lambda_n^{3/2} C_g$ *for* $n \geq N_0$,

in which C_f *and* C_g *are positive constants independent of* n. *Then*

$$\sum_{n=N_0}^{\infty} (1 + \lambda_n^2)(f_n^2 + \lambda_n^{-2} g_n^2) \leq (C_f^0 + C_g^0) \sum_{n=N_0}^{\infty} (1 + \lambda_n^2) \lambda_n^{-3}$$

in which

$$C_f^0 \leq C \| L[f] \|_{L^2} \lambda_1^{-1/2} + C_f,$$

$$C_g^0 \leq C \| L[g] \|_{L^2} \lambda_1^{-3/2} + C_g.$$

Proof. From (34) and the assumptions (2) and (3) it follows that

$$|f_n| \leq \lambda_n^{-3/2} \left[C \| L[f] \|_{L^2} \lambda_n^{-1/2} + C_f \right]$$

and

$$|g_n| \leq \lambda_n^{-1/2} \left[C \| L[g] \|_{L^2} \lambda_n^{-3/2} + C_g \right].$$

Hence

$$f_n^2 + \lambda_n^{-2} g_n^2 \leq \lambda_n^{-3} (C_f^0 + C_g^0) \qquad \text{for } n \geq N_0$$

and the result follows.

EXAMPLES. $L[w] = w_{xxxx}$, $l = 1$, $T = 1$, and $m(x) = 1$.

(a) Simply supported, $\Omega_n = n\pi$, $n = 1, 2, 3, \ldots$,

$$w(0,t) = w_{xx}(0,t) = 0, \qquad w(1,t) = w_{xx}(1,t) = 0.$$

(b) Clamped at both ends, $\Omega_n = (n + \frac{1}{2})\pi$, $n \geq 4$ (for $n = 1, 2, 3$ see Magrab [5]),

$$w(0, t) = w_x(0, t) = 0, \qquad w(1, t) = w_x(1, t) = 0.$$

(c) Cantilevered, $\Omega_n = (n - \frac{1}{2})\pi$, $n \geq 4$ (for $n = 1, 2, 3$ see Magrab [5]),

$$w(0, t) = w_x(0, t) = 0, \qquad w_{xx}(1, t) = w_{xxx}(1, t) = 0.$$

These problems are self-adjoint and hence the boundary conditions that f and g are assumed to satisfy in Theorem 4 are the same as those on w.

Consider the case of initial conditions

$$w(x, 0) = f(x) = x^3(1 - x)^4,$$
$$w_t(x, 0) = g(x) = \sin^4 \pi x$$

which are eligible for all of the above cases since

$$f(0) = f'(0) = f''(0) = 0 \quad \text{and} \quad f(1) = f'(1) = f''(1) = f'''(1) = 0$$

and similarly for g. Moreover,

$$\|L[f]\|_{L^2}^2 = \|f^{IV}\|_{L^2}^2 = \int_0^1 \left[f^{IV}(x)\right]^2 dx = 576,$$

$$\|L[g]\|_{L^2}^2 = \|g^{IV}\|_{L^2}^2 = \int_0^1 \left[g^{IV}(x)\right]^2 dx = 0.516176 \times 10^7.$$

Figures 1 and 2 can be used to determine the number of quenching terms needed to insure that

$$\int_0^1 \left[w^2(x, t; u_N) + w_t^2(x, t; u_N)\right] dx < \varepsilon \qquad t \geq T = 1$$

when the control is turned off at $t = 1$. Directly from the figure it can be seen that, for example, if an error of $\frac{\varepsilon}{2} < 2 \times 10^{-2}$ is required, then $N = 4$ quenching terms are sufficient and if an error of $\frac{\varepsilon}{2} < 10^{-6}$ is required, then 50 quenching terms are sufficient.

EXAMPLE. Consider a cantilever beam described by

$$w_{tt} + w_{xxxx} = u(x, t), \qquad \text{on } R = \{(x, t) \mid 0 < x < 1, 0 < t < 1\} \tag{35}$$

with boundary conditions

$$w(0, t) = 0, \qquad w_x(0, t) = 0 \tag{36}$$

$$w_{xx}(1, t) = 0, \qquad w_{xxx}(1, t) = 0 \tag{37}$$

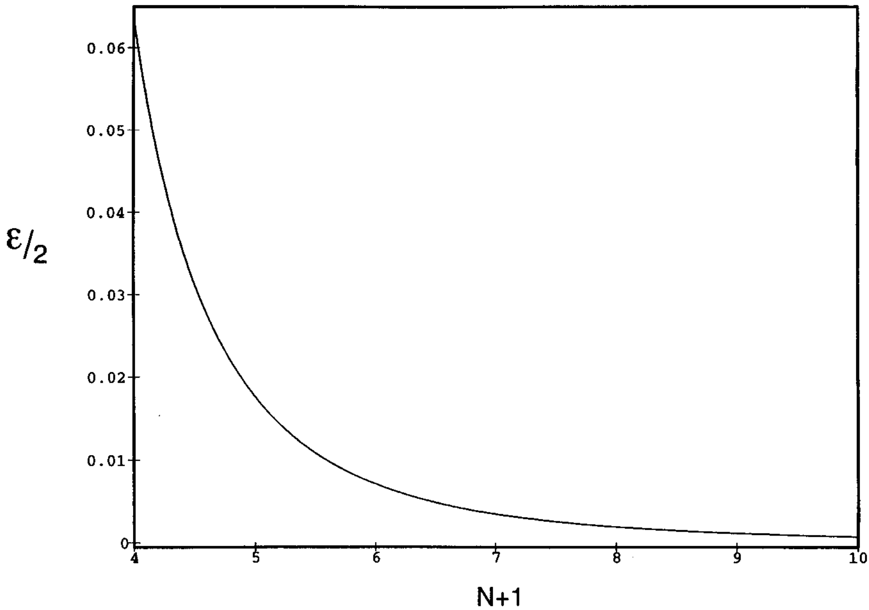

FIG. 1. Residual error versus number of quenching terms.

and initial conditions

$$w(x,0) = f(x) = x^2(1-x)^2, \tag{38}$$

$$w_t(x,0) = g(x) = 0. \tag{39}$$

The spatial eigenvalue problem becomes

$$\varphi_n^{IV}(x) = \lambda_n^2 \varphi_n(x) \tag{40}$$

$$\varphi_n(0) = \varphi_n'(0) = 0, \qquad \varphi_n''(1) = \varphi'''(1) = 0. \tag{41}$$

The eigenvalues are given as roots of the characteristic equation

$$\cos \Omega_n \cosh \Omega_n = -1, \qquad \lambda_n = \Omega_n^2 > 0 \tag{42}$$

in which the Ω_n are the natural frequencies. Eigenfunctions $\tilde{\varphi}_n(x)$ are given by

$$\tilde{\varphi}_n(x) = D_n T(\Omega_n x) + S(\Omega_n x), \tag{43}$$

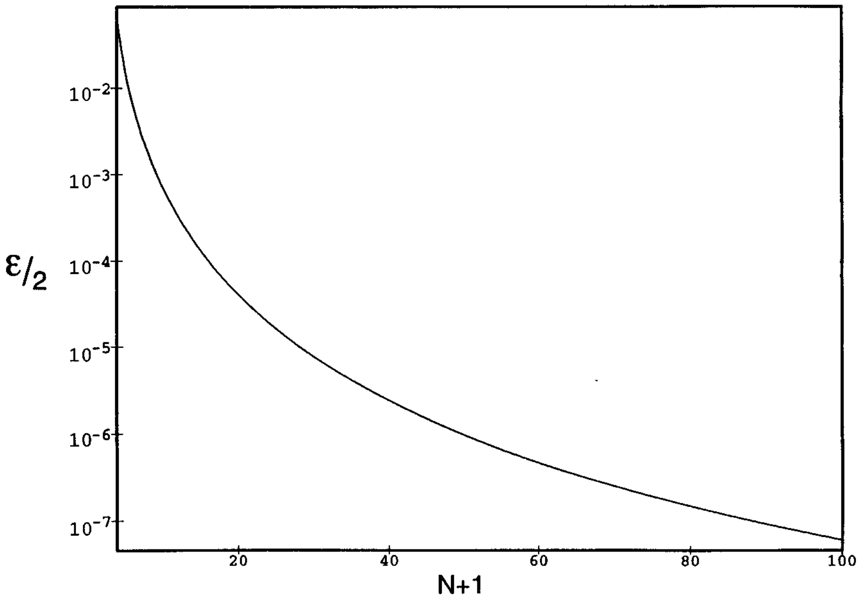

FIG. 2. Residual error versus number of quenching terms.

where

$$D_n = -T(\Omega_n)/U(\Omega_n)$$

in which

$$U(x) = \tfrac{1}{2}[\cosh x + \cos x],$$

$$S(x) = \tfrac{1}{2}[\cosh x - \cos x],$$

$$T(x) = \tfrac{1}{2}[\sinh x - \sin x].$$

Since $\|\tilde{\varphi}_n(x)\|_{L^2} = \tilde{\varphi}_n(1)$, orthonormal eigenfunctions are given by

$$\varphi_n(x) = \frac{\tilde{\varphi}_n(x)}{\tilde{\varphi}_n(1)}.$$

In this example

$$U_{\mathrm{ad}}^0 = \{u \mid u(x,t) \in L^2[0,1; L^2(0,1)]\}$$

and

$$U_{\mathrm{ad}} = \{u \mid u \in U_{\mathrm{ad}}^0 \text{ and } u \text{ is a quencher for } w(x,t;u) \text{ at } t = 1\}.$$

For quenching we seek a $u(x,t) \in U_{ad}$ such that $w(x,t;u)$ satisfies (35)–(39) as well as

$$w(x,t;u) = w_t(x,t;u) = 0 \qquad \text{at } t = 1, 0 \le x \le 1.$$

In this example,

$$\langle \alpha, \beta \rangle = \int_0^1 \alpha(x)\beta(x)\,dx, \qquad \|\alpha\|^2 = \int_0^1 \alpha^2(x)\,dx,$$

$$w(x,t) = \sum_{n=1}^{\infty} z_n(t)\varphi_n(x) \tag{44}$$

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t)\varphi_n(x) \tag{45}$$

$$f(x) = \sum_{n=1}^{\infty} f_n\varphi_n(x), \qquad f_n = \langle f, \varphi_n \rangle / \|\varphi_n\|^2 \tag{46}$$

$$g(x) = \sum_{n=1}^{\infty} g_n\varphi_n(x), \qquad g_n = 0. \tag{47}$$

Note that $f(x)$ does not satisfy the homogeneous boundary conditions and thus we must investigate the applicability of the corollary of Theorem 4. Note that

$$\Omega_n = \lambda_n^{1/2} = \left(n - \tfrac{1}{2}\right)\pi \qquad \text{for } n \ge 4.$$

Also note that for all n

$$\begin{aligned} \beta[f, \varphi_n]_0^1 &= f_{xx}(1)\varphi_{n,x}(1) - f_{xxx}(1)\varphi_n(1) \\ &= 2\varphi_{n,x}(1) - 12\varphi_n(1) \end{aligned}$$

and that

$$\beta[g, \varphi_n]_0^1 = 0.$$

Consider

$$\varphi_{n,x}(\Omega_n) = \Omega_n[D_n S(\Omega_n) + V(\Omega_n)],$$

where

$$V(x) = \tfrac{1}{2}[\sinh x + \sin x].$$

It follows that

$$\varphi_{n,x}(\Omega_n) = \Omega_n \frac{-T(\Omega_n)S(\Omega) + U(\Omega_n)V(\Omega_n)}{U(\Omega_n)}$$

$$= \left(\frac{\Omega_n}{2}\right) \frac{\begin{array}{c}-(\sinh \Omega - \sin \Omega_n)(\cosh \Omega_n - \cos \Omega_n)\\ +(\cosh \Omega_n + \cos \Omega_n)(\sinh \Omega_n + \sin \Omega_n)\end{array}}{\cosh \Omega_n + \cos \Omega_n}$$

$$= \left(\frac{\Omega_n}{2}\right)\left[\frac{-\sin \Omega_n \cos \Omega_n + \cosh \Omega_n \sin \Omega_n + \sinh \Omega_n \cos \Omega_n}{\cosh \Omega_n + \cos \Omega_n}\right.$$

$$\left. + \frac{\cosh \Omega_n \sin \Omega_n + \sinh \Omega_n \cos \Omega_n + \cos \Omega_n \sin \Omega_n}{\cosh \Omega_n + \cos \Omega_n}\right]$$

$$= \Omega_n \frac{\cosh \Omega_n \sin \Omega_n + \sinh \Omega_n \cos \Omega_n}{\cosh \Omega_n + \cos \Omega_n}$$

$$= \Omega_n \frac{(e^{\Omega_n} + e^{-\Omega_n})\sin \Omega_n + (e^{\Omega_n} - e^{-\Omega_n})\cos \Omega_n}{(e^{\Omega_n} + e^{-\Omega_n}) + 2\cos \Omega_n}$$

$$= \Omega_n \frac{\left[\sin \Omega_n + \cos \Omega_n + e^{-2\Omega_n}(\sin \Omega_n - \cos \Omega_n)\right]}{1 + \eta_n}$$

in which $\eta_n = e^{-2\Omega_n} + 2e^{-\Omega_n} \cos \Omega_n$.

Hence

$$|\varphi_{n,x}(1)| \le 2\Omega_n\left[(1 + e^{-2\Omega_1})/(1 - 2e^{-\Omega_1})\right]$$
$$\le 2\Omega_n(1.477)$$

and

$$\left|\beta[f, \varphi_n]\big|_0^1\right| \le 4(1.477)\Omega_n + 12$$
$$\le C_f \Omega_n$$

in which

$$C_f = 4(1.477) + 12 \qquad \text{for all } n \ge 1$$

and the conditions of the corollary are satisfied since $C_g = 0$.

It follows since

$$\lambda_n = \Omega_n^2 = \left(n - \frac{1}{2}\right)^2 \pi^2 \qquad \text{for } n \geq 4$$

that

$$\sum_{n=4}^{\infty} \left(1 + \lambda_n^2\right)\lambda_n^{-3} \leq \sum_{n=4}^{\infty} \left(1 + \Omega_n^4\right)\Omega_n^{-6} < \infty$$

and hence Theorem 3 can be used.

The quenching problem becomes one of solving

$$\ddot{z}_n(t) + \lambda_n^2 z_n(t) = u_n(t), \qquad 0 < t < 1 \tag{48}$$

with

$$z_n(0) = f_n \quad \text{and} \quad \dot{z}_n(0) = g_n = 0, \tag{49}$$

$$z_n(1) = 0 \quad \text{and} \quad \dot{z}_n(1) = 0. \tag{50}$$

The coefficient of the nth mode of the optimal quencher will be of the form

$$u_n(t) = \sigma_n \cos \lambda_n t + \mu_n \sin \lambda_n t. \tag{51}$$

The general solution of the equation is given by

$$z_n(t) = z_n^h(t) + z_n^p(t), \tag{52}$$

where

$$z_n^h(t) = f_n \cos \lambda_n t + (g_n/\lambda_n)\sin \lambda_n t, \tag{53}$$

$$z_n^p(t) = z_n^p(t; \sigma_n, \mu_n) = \lambda_n^{-1}\int_0^t u_n(s)\sin \lambda_n(t - s)\, ds. \tag{54}$$

The conditions for quenching, (50), become

$$z_n(1; \sigma_n, \mu_n) = z_n^h(1) + z_n^p(1; \sigma_n, \mu_n) = 0, \tag{55}$$

$$\dot{z}_n(1; \sigma_n, \mu_n) = \dot{z}_n^h(1) + \dot{z}_n^p(1; \sigma_n, \mu_n) = 0 \tag{56}$$

which are two linear equations in the two unknowns σ_n and μ_n.

5. NUMERICAL RESULTS

The numerical results were obtain using Maple V software. For the first six modes

Mode n	Ω_n	σ_n	μ_n
1	0.59684π	-0.013031	0.375687
2	1.494176π	-0.000237	2.781357
3	2.500247π	-0.011658	0.870054
4	3.499989π	0.004264	-0.516963
5	4.500000π	-0.003133	0.718191
6	5.500000π	0.000021	-0.569004

Figures 3–8 demonstrate how modes 1, 3, and 6 are being quenched for the time part of the displacement $z(t)$ and velocity $\dot{z}(t)$.

6. SUMMARY

An Euler–Bernoulli beam is considered with linear homogeneous boundary conditions and initial conditions. With terminal time T prescribed, the problem of quenching the vibration of the beam at time T

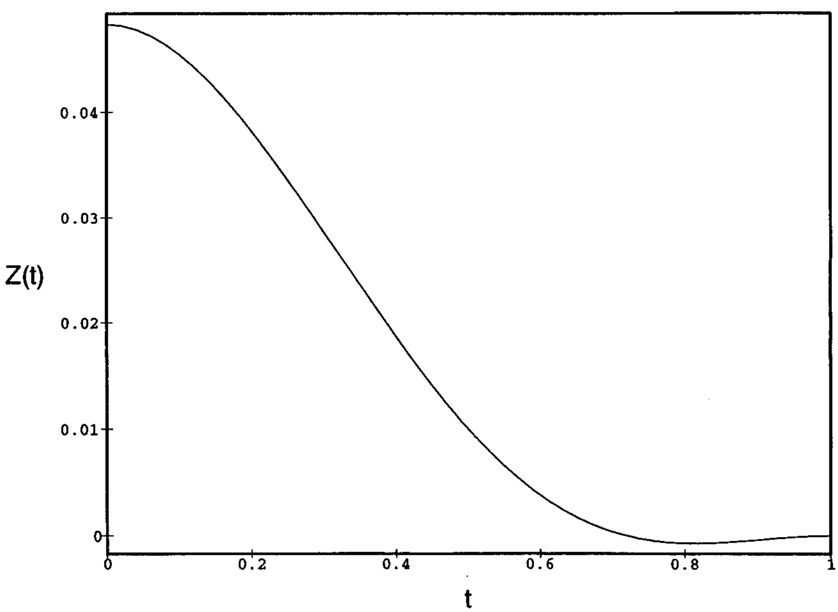

FIG. 3. Time part of the displacement for mode 1 quenched at $T = 1$.

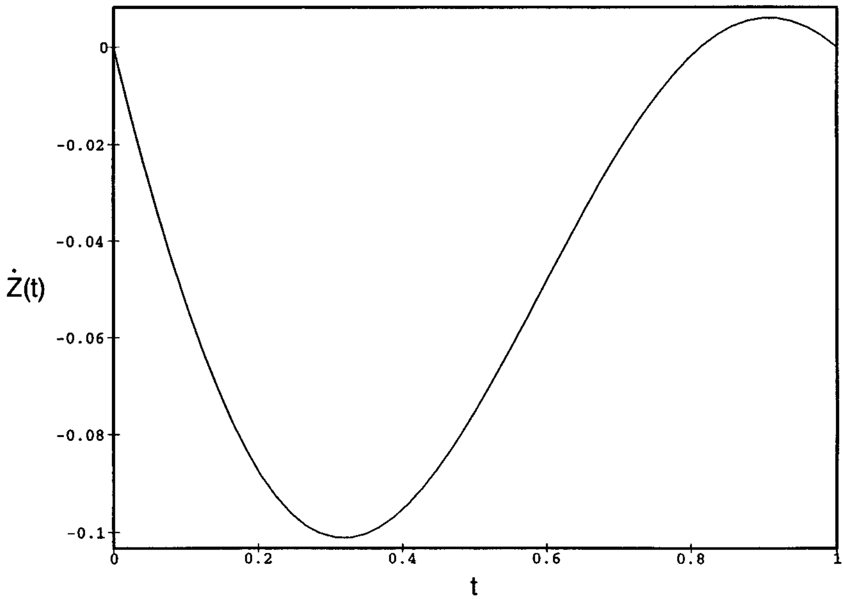

FIG. 4. Time part of the velocity for mode 1 quenched at $T = 1$.

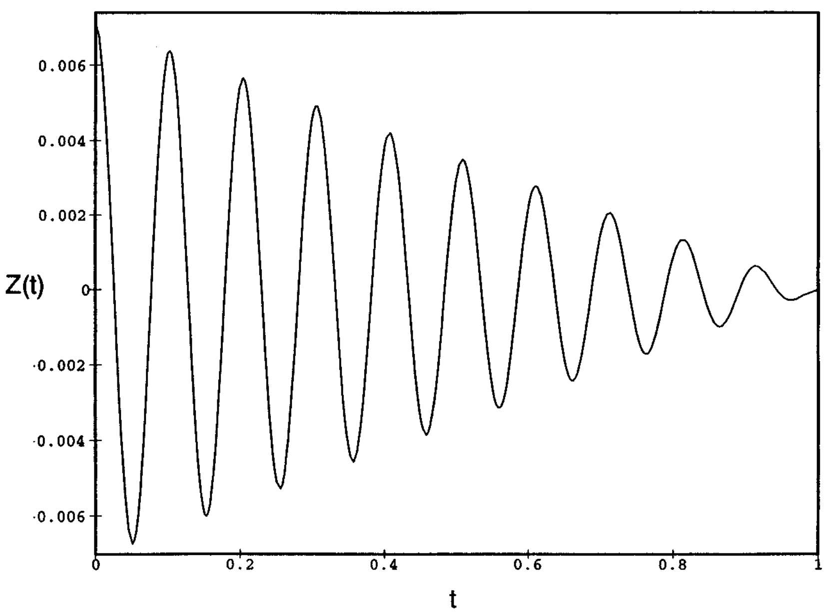

FIG. 5. Time part of the displacement for mode 3 quenched at $T = 1$.

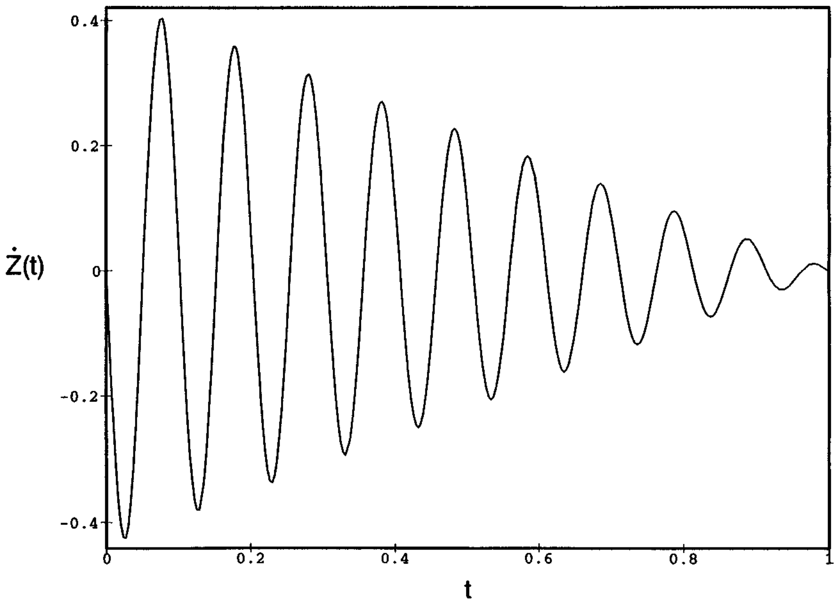

FIG. 6. Time part of the velocity for mode 3 quenched at $T = 1$.

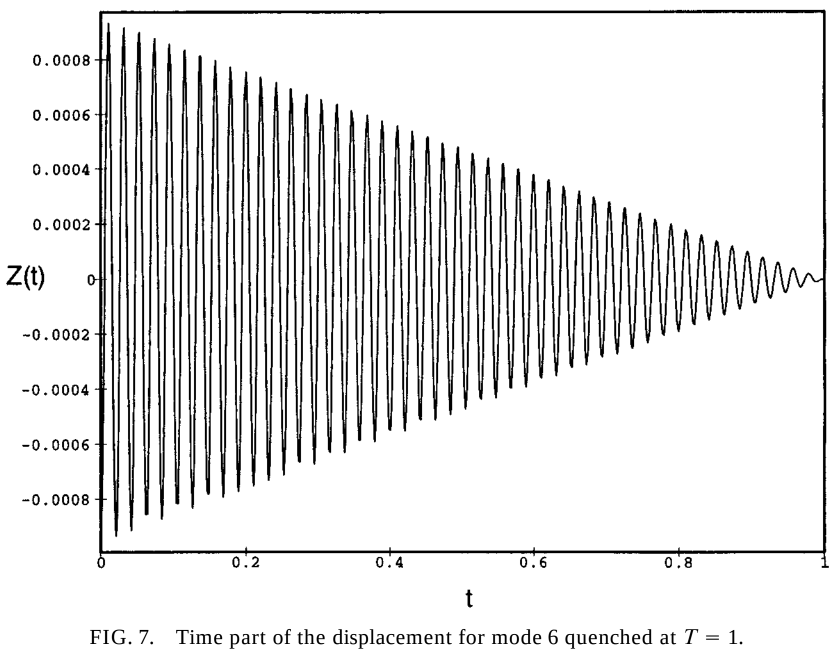

FIG. 7. Time part of the displacement for mode 6 quenched at $T = 1$.

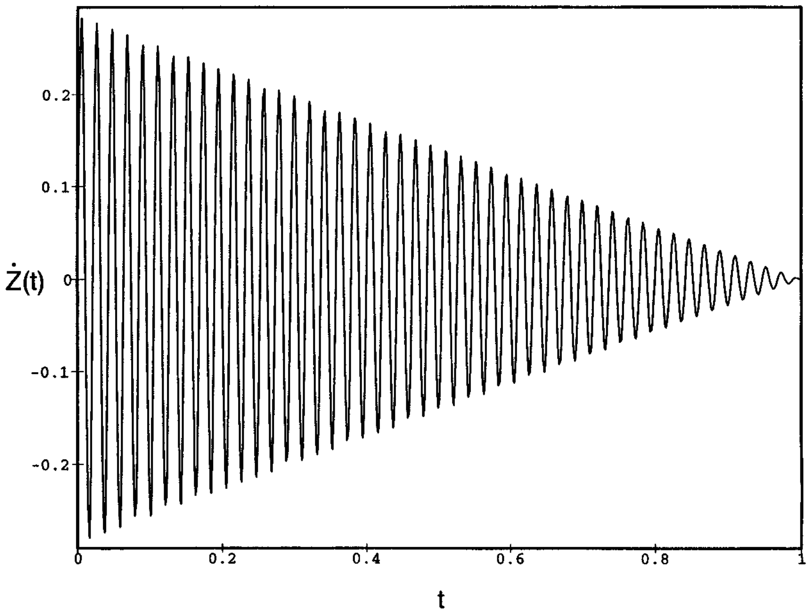

FIG. 8. Time part of the velocity for mode 6 quenched at $T = 1$.

with an applied force control, i.e., the problem of bringing the vibration to a stop at time T, is addressed. An example is given that shows quenchers are in general not unique. If in addition to requiring the applied force to quench the vibration at time T, we require moreover that the least amount of force be used, then the optimal quenching force will be unique.

For one mode an explicit expression for the control is given that will quench the motion at time $t = T$. It is observed that this given quenching force uses the least amount of force from among all quenchers. Under reasonable conditions, it is shown that the sequence of modal quenchers can be used to construct a single distributed control force that will quench each mode. Moreover this quencher is the unique control that utilizes the minimum amount of force.

In practice, only a finite number of modes can be quenched at terminal time T. Without further conditions an example is given that shows no matter how many preassigned modes N are quenched, the solution can deviate from 0 for $t \geq T$ by a very large amount. If at terminal time, the control is turned off, then it is shown that under certain conditions the dynamic response can be made smaller than any preassigned amount.

Examples illustrating that the conditions are satisfied for a beam that is simply supported, or clamped at both ends, or cantilevered are given.

A numerical example is given to illustrate the results.

REFERENCES

1. S. Jayasuriya and S. Choura, Active quenching of a set of predetermined vibratory modes of a beam by a single fixed point actuator, *Internat. J. Control* **51**, No. 2 (1990), 445–467.
2. T. A. Trautt, ‘‘Inverse Dynamics of Flexible Articulated Structures with Non-Zero Initial Conditions,’’ Ph.D. dissertation, Dept. of Mechanical and Environmental Engineering, University of California, Santa Barbara, 1996.
3. C. C. Kao, ‘‘Vibration Quenching with General Boundary Conditions,’’ M.A. thesis, Dept. of Mathematics, University of California, Santa Barbara, 1993.
4. S. Barnett and R. G. Cameron, ‘‘Introduction to Mathematical Control Theory,’’ 2nd ed., Clarendon, Oxford, 1990.
5. E. B. Magrab, ‘‘Vibration of Elastic Structural Members,’’ Sijthoff and Noordhoff, Alphen aan den Rijn, The Netherlands, 1979.